Numerical algorithms for uniform Airy-type asymptotic expansions

N.M. Temme

CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
E-mail: nicot@cwi.nl

Received 16 December 1996; revised 20 March 1997
Communicated by S. Paszkowski

Airy-type asymptotic representations of a class of special functions are considered from a numerical point of view. It is well known that the evaluation of the coefficients of the asymptotic series near the transition point is a difficult problem. We discuss two methods for computing the asymptotic series. One method is based on expanding the coefficients of the asymptotic series in Maclaurin series. In the second method we consider auxiliary functions that can be computed more efficiently than the coefficients in the first method, and we do not need the tabulation of many coefficients. The methods are quite general, but the paper concentrates on Bessel functions, in particular on the differential equation of the Bessel functions, which has a turning point character when order and argument of the Bessel functions are equal.

Keywords: uniform asymptotic expansions, turning points, Airy-type expansions, Bessel functions, computation of special functions

AMS subject classification: 41A60, 34E20, 33C10, 65D20

1. Introduction

Writing efficient algorithms for special functions may become problematic when several large parameters are involved. In particular problems arise when functions suddenly change their behaviour, say from monotonic to oscillatory behaviour. For many special functions of mathematical physics powerful uniform asymptotic expansions are available, which describe precisely how the functions behave, which are valid for large domains of the parameters, and which provide tools for designing high-performance computational algorithms. An important class concerns the functions having a turning point in their defining differential equation, in which case Airy-type expansions arise.

Airy functions are solutions of the differential equation

$$\frac{d^2 w}{d z^2} = zw. \tag{1.1}$$

Two linearly independent solutions that are real for real values of z are denoted by $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$. Equation (1.1) is the simplest second order linear differential equation that has a simple turning point (at $z = 0$). More general turning point equations have the standard from

$$\frac{\mathrm{d}^2 w}{\mathrm{d}\zeta^2} = \left[u^2 \zeta + \psi(\zeta)\right] w \tag{1.2}$$

and the problem is to find an asymptotic approximation of $w(\zeta)$ for large values of u, that holds uniformly in a neighborhood of $\zeta = 0$. A first approximation is obtained by neglecting $\psi(\zeta)$, which gives the solutions

$$\mathrm{Ai}\left(u^{2/3}\zeta\right), \qquad \mathrm{Bi}\left(u^{2/3}\zeta\right).$$

For a detailed discussion of this kind of problems we refer to [7, chapter 11]. Many physical problems and special functions can be transformed into the standard form (1.2). Examples are Bessel functions, Whittaker functions, the classical orthogonal polynomials (in particular Hermite and Laguerre polynomials), and parabolic cylinder functions. The existing uniform expansions for all these functions are powerful in an analytic sense. In several cases rigorous and realistic bounds are given for the remainders of the expansions; cf. [7].

From a computational point of view the uniform character of the expansions causes a difficulty. This is mainly due to the complexity of the coefficients in the expansions. In all known cases the coefficients are difficult to compute in the neighborhood of the turning point. Usually this point is of special interest in the algorithms, since many other methods fail in the turning point area when the parameters are large. In [2] uniform Airy-type expansions are used for the evaluation of Bessel functions. Matviyenko [5] discusses the implementation of several kinds of asymptotic expansions of the Bessel functions. However, Matviyenko does not use Airy-type expansions. For the turning point region he proposes numerical quadrature for the Sommerfeld integral of the Hankel functions, after selecting contours of steepest descents. It is of interest to compare the algorithms of Amos and Matviyenko with our algorithms, but we expect to return to this in future publications, when we also want to consider the modified Bessel functions with purely imaginary order (cf. [3] and [9]).

In this paper we discuss two methods for computing the asymptotic series. One method is based on expanding the coefficients in the series into Maclaurin series. We show how to obtain the coefficients of the Maclaurin series for the coefficients of the asymptotic series. In the second method we consider auxiliary functions that can be computed more efficiently than the coefficients in the first method; in addition, we do not need the tabulation of many coefficients. In fact we consider differential equations for functions representing (in an exact sense) the asymptotic series, and we base a numerical algorithm directly on these differential equations. Extra features of the second method are:

- we deal with *convergent* expansions;
- we need only a small number of pre-computed tabulated numbers;

- the method is applicable for quite small values of the large parameter.

In some sense this method is similar to the one described for the computation of incomplete gamma functions in [8]. In that case the error function is the main approximant.

The methods described in this paper are quite general, but we only treat the case of Bessel functions, by using the differential equation of the Bessel functions, which has a turning point character when order and argument of the Bessel functions are equal.

In the following section we summarize the Airy-type expansions for the Bessel functions and their derivatives. In section 3 we describe the method of obtaining Maclaurin series expansions for the coefficients in the expansions. In section 4 we describe a second method based on an iteration scheme to compute auxiliary functions that replace the asymptotic series. In a final section we give details of numerical experiments.

2. Airy-type asymptotics of ordinary Bessel functions

The ordinary Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ can be expanded in terms of Airy functions. From [1, p. 368] and [7, p. 425] we obtain the following results,

$$\begin{aligned} J_\nu(\nu z) &= \frac{\phi(\zeta)}{\nu^{1/3}}\left[\mathrm{Ai}\left(\nu^{2/3}\zeta\right)A_\nu(\zeta) + \nu^{-4/3}\mathrm{Ai}'\left(\nu^{2/3}\zeta\right)B_\nu(\zeta)\right], \\ Y_\nu(\nu z) &= -\frac{\phi(\zeta)}{\nu^{1/3}}\left[\mathrm{Bi}\left(\nu^{2/3}\zeta\right)A_\nu(\zeta) + \nu^{-4/3}\mathrm{Bi}'\left(\nu^{2/3}\zeta\right)B_\nu(\zeta)\right], \end{aligned} \tag{2.1}$$

where

$$A_\nu(\zeta) \sim \sum_{s=0}^{\infty} \frac{a_s(\zeta)}{\nu^{2s}}, \qquad B_\nu(\zeta) \sim \sum_{s=0}^{\infty} \frac{b_s(\zeta)}{\nu^{2s}} \tag{2.2}$$

as $\nu \to \infty$, uniformly with respect to $z \in [0, \infty)$. The expansions are valid for complex values of ν and z, but here we concentrate on real values of the parameters.

The parameter ζ is defined by

$$\begin{aligned} \frac{2}{3}\zeta^{3/2} &= \ln\frac{1+\sqrt{1-z^2}}{z} - \sqrt{1-z^2}, \quad 0 \leqslant z \leqslant 1, \\ \frac{2}{3}(-\zeta)^{3/2} &= \sqrt{z^2-1} - \arccos\frac{1}{z}, \quad z \geqslant 1. \end{aligned} \tag{2.3}$$

Furthermore,

$$\phi(\zeta) = \left(\frac{4\zeta}{1-z^2}\right)^{1/4}, \quad \phi(0) = 2^{1/3}. \tag{2.4}$$

The first coefficients a_s, b_s are

$$a_0(\zeta) = 1, \qquad b_0(\zeta) = -\frac{5}{48\zeta^2} + \frac{\phi^2(\zeta)}{48\zeta}\left[\frac{5}{1-z^2} - 3\right].$$

Higher coefficients follow from the representations

$$a_s(\zeta)=\sum_{k=0}^{2s}\mu_k\zeta^{-3k/2}u_{2s-k}(t),$$
$$b_s(\zeta)=-\zeta^{-1/2}\sum_{k=0}^{2s+1}\lambda_k\zeta^{-3k/2}u_{2s+1-k}(t), \qquad (2.5)$$

where $t=1/\sqrt{1-z^2}$, $\lambda_0=\mu_0=1$,

$$\lambda_k=\frac{(2k+1)(2k+3)\cdots(6k-1)}{k!(144)^k}, \qquad \mu_k=-\frac{6k+1}{6k-1}\lambda_k, \quad k=1,2,3,\ldots. \qquad (2.6)$$

The quantities u_k are given by

$$u_{k+1}(t)=\tfrac{1}{2}t^2\left(1-t^2\right)u_k'(t)+\frac{1}{8}\int_0^t\left(1-5\tau^2\right)u_k(\tau)\,\mathrm{d}\tau, \quad k=0,1,2,\ldots, \qquad (2.7)$$

with $u_0(t)=1$.

Asymptotic representations for the Hankel functions follow from the relations

$$H_\nu^{(1)}(z)=J_\nu(z)+iY_\nu(z), \qquad H_\nu^{(2)}(z)=J_\nu(z)-iY_\nu(z),$$

and

$$\mathrm{Ai}(z)+i\mathrm{Bi}(z)=2\mathrm{e}^{\pi i/3}\mathrm{Ai}\left(z\mathrm{e}^{-2\pi i/3}\right), \qquad \mathrm{Ai}(z)-i\mathrm{Bi}(z)=2\mathrm{e}^{-\pi i/3}\mathrm{Ai}\left(z\mathrm{e}^{2\pi i/3}\right).$$

This gives representations for the Hankel functions with the same structure as for the ordinary Bessel functions, with the same functions $A_\nu(\zeta)$, $B_\nu(\zeta)$.

2.1. *Representations for the derivatives*

For the derivatives we have (cf. [1, p. 369])

$$J_\nu'(\nu z)=-\widehat{\phi}(\zeta)\left[\nu^{-4/3}\mathrm{Ai}\left(\nu^{2/3}\zeta\right)C_\nu(\zeta)+\nu^{-2/3}\mathrm{Ai}'\left(\nu^{2/3}\zeta\right)D_\nu(\zeta)\right],$$
$$Y_\nu'(\nu z)=\widehat{\phi}(\zeta)\left[\nu^{-4/3}\mathrm{Bi}\left(\nu^{2/3}\zeta\right)C_\nu(\zeta)+\nu^{-2/3}\mathrm{Bi}'\left(\nu^{2/3}\zeta\right)D_\nu(\zeta)\right], \qquad (2.8)$$

where

$$\widehat{\phi}(\zeta)=-\frac{\mathrm{d}\zeta}{\mathrm{d}z}\phi(\zeta)=\frac{2}{z\phi(\zeta)},$$

$$C_\nu(\zeta)=\chi(\zeta)A_\nu(\zeta)+A_\nu'(\zeta)+\zeta B_\nu(\zeta),$$
$$D_\nu(\zeta)=A_\nu(\zeta)+\nu^{-2}\chi(\zeta)B_\nu(\zeta)+\nu^{-2}B_\nu'(\zeta), \qquad (2.9)$$

$$\chi(\zeta)=\frac{\phi'(\zeta)}{\phi(\zeta)}=\frac{4-z^2[\phi(\zeta)]^6}{16\zeta}.$$

Primes denote differentiation with respect to ζ.

The functions $C_\nu(\zeta)$, $D_\nu(z)$ have the expansions

$$C_\nu(\zeta) \sim \sum_{s=0}^{\infty} \frac{c_s(\zeta)}{\nu^{2s}}, \qquad D_\nu(\zeta) \sim \sum_{s=0}^{\infty} \frac{d_s(\zeta)}{\nu^{2s}}, \tag{2.10}$$

where

$$\begin{aligned} c_s(\zeta) &= \chi(\zeta)a_s(\zeta) + a_s'(\zeta) + \zeta b_s(\zeta), \\ d_s(\zeta) &= a_s(\zeta) + \chi(\zeta)b_{s-1}(\zeta) + b_{s-1}'(\zeta). \end{aligned} \tag{2.11}$$

The first coefficients c_s, d_s are

$$c_0(\zeta) = \frac{7}{48\zeta} + \frac{\phi^2(\zeta)}{48}\left[9 - \frac{7}{1-z^2}\right], \qquad d_0(\zeta) = 1.$$

Higher coefficients follow from the representations

$$\begin{aligned} c_s(\zeta) &= -\zeta^{1/2} \sum_{k=0}^{2s+1} \mu_k \zeta^{-3k/2} v_{2s+1-k}(t), \\ d_s(\zeta) &= \sum_{k=0}^{2s} \lambda_k \zeta^{-3k/2} v_{2s-k}(t), \end{aligned} \tag{2.12}$$

where t, λ_k and μ_k are as in (2.5)–(2.7), and the quantities v_k can be expressed in terms of the u_k of (2.7):

$$v_k(t) = u_k(t) + t(t^2-1)\left[\tfrac{1}{2}u_{k-1}(t) + t u_{k-1}'(t)\right], \quad k = 1, 2, \ldots,$$

with $v_0(t) = 1$.

Explicit representations of $a_s'(\zeta)$, $b_s'(\sigma)$ can be obtained by differentiating the relations in (2.5), but they also follow from the representations for a_s, b_s, c_s, d_s and from (2.11):

$$\begin{aligned} a_s'(\zeta) &= c_s(\zeta) - \chi(\zeta)a_s(\zeta) - \zeta b_s(\zeta), \\ b_s'(\zeta) &= d_{s+1}(\zeta) - a_{s+1}(\zeta) - \chi(\zeta)b_s(\zeta). \end{aligned} \tag{2.13}$$

A recursive scheme for evaluating a_s, b_s is given by

$$\begin{aligned} a_s''(\zeta) + 2\zeta b_s'(\zeta) + b_s(\zeta) - \psi(\zeta)a_s(\zeta) &= 0, \\ 2a_{s+1}'(\zeta) + b_s''(\zeta) - \psi(\zeta)b_s(\zeta) &= 0, \end{aligned} \tag{2.14}$$

where $a_0(\zeta) = 1$ and

$$\psi(\zeta) = \frac{5}{16\zeta^2} + \frac{\zeta z^2(z^2+4)}{4(z^2-1)^3}. \tag{2.15}$$

The coefficients a_s, b_s, c_s, d_s in (2.2) and (2.10) are complicated expressions. Explicit representations are given in (2.5) and (2.12) in terms of the coefficients u_k of Debye-type asymptotic expansions. However, these expressions are difficult to compute near the *turning point* $z = 1$, or equivalently, near $\zeta = 0$. In [2] all needed coefficients a_s, b_s are expanded in Maclaurin series at the turning point, the Maclaurin series being in terms of the variable $w^2 = 1 - z^2$.

2.2. *Further properties of the functions* $A_\nu, B_\nu, C_\nu, D_\nu$

Using the Wronskian relation for the Airy functions, viz.

$$\mathrm{Ai}(z)\mathrm{Bi}'(z) - \mathrm{Ai}'(z)\mathrm{Bi}(z) = \frac{1}{\pi}, \tag{2.16}$$

we can invert the relations in (2.1) and obtain

$$\begin{aligned} A_\nu(\zeta) &= \frac{\pi\nu^{1/3}}{\phi(\zeta)}\left[J_\nu(\nu z)\mathrm{Bi}'\left(\nu^{2/3}\zeta\right) + Y_\nu(\nu z)\mathrm{Ai}'\left(\nu^{2/3}\zeta\right)\right], \\ B_\nu(\zeta) &= -\frac{\pi\nu^{5/3}}{\phi(\zeta)}\left[J_\nu(\nu z)\mathrm{Bi}\left(\nu^{2/3}\zeta\right) + Y_\nu(\nu z)\mathrm{Ai}\left(\nu^{2/3}\zeta\right)\right]. \end{aligned} \tag{2.17}$$

The functions $A_\nu(\zeta)$ and $B_\nu(\zeta)$ are the 'slowly varying' parts in the representations in (2.1).

Olver's approach for deriving Airy-type expansions for the Bessel functions is based on the differential equation

$$\frac{\mathrm{d}^2W}{\mathrm{d}\zeta^2} = \left[\nu^2\zeta + \psi(\zeta)\right]W, \tag{2.18}$$

where $\psi(\zeta)$ is given in (2.15). This differential equation is obtained from the well-known Bessel equation by using a Liouville–Green transformation; see [7, p. 420]. The quantities within the square brackets in (2.1) are two solutions of equation (2.18).

By using equation (2.18), we can derive the following system of differential equations for the functions $A_\nu(\zeta), B_\nu(\zeta)$:

$$\begin{aligned} A'' + 2\zeta B' + B - \psi(\zeta)A &= 0, \\ B'' + 2\nu^2 A' - \psi(\zeta)B &= 0, \end{aligned} \tag{2.19}$$

where primes denote differentiation with respect to ζ. To verify this we write equation (2.18) in the operator form $\mathcal{L}_\zeta W(\zeta) = 0$. Applying $\mathcal{L}_\zeta$ to

$$W(\zeta) = \mathrm{Ai}\left(\nu^{2/3}\zeta\right)A_\nu(\zeta) + \nu^{-4/3}\mathrm{Ai}'\left(\nu^{2/3}\zeta\right)B_\nu(\zeta),$$

we find

$$\begin{aligned} \mathcal{L}_\zeta W(\zeta) = {} & \mathrm{Ai}\left(\nu^{2/3}\zeta\right)\left[A_\nu''(\zeta) + 2\zeta B_\nu'(\zeta) + B_\nu(\zeta) - \psi(\zeta)A_\nu(\zeta)\right] \\ & + \nu^{-4/3}\mathrm{Ai}'\left(\nu^{2/3}\zeta\right)\left[B_\nu''(\zeta) + 2\nu^2 A_\nu'(\zeta) - \psi(\zeta)B_\nu(\zeta)\right], \end{aligned} \tag{2.20}$$

where we have used the differential equation of the Airy functions; cf. (1.1). Because $\mathcal{L}_\zeta W(\zeta) \equiv 0$, the quantities within square brackets in (2.20) must vanish.

A Wronskian for the system (2.19) follows by eliminating the terms with $\psi(\zeta)$. This gives

$$A''B - B''A + B^2 + 2\zeta B'B - 2\nu^2 A'A = 0,$$

which can be integrated:

$$\nu^2 A_\nu^2(\zeta) + A_\nu(\zeta)B_\nu'(\zeta) - A_\nu'(\zeta)B_\nu(\zeta) - \zeta B_\nu^2(\zeta) = \nu^2. \tag{2.21}$$

The constant on the right-hand side follows by taking $\zeta = 0$ and from information given later in this section.

By using the Wronskian relation for the Bessel functions:

$$J_\nu(z)Y_\nu'(z) - J_\nu'(z)Y_\nu(z) = \frac{2}{\pi z}, \tag{2.22}$$

it follows that $A_\nu, B_\nu, C_\nu, D_\nu$ are related in the following way:

$$A_\nu(\zeta)D_\nu(\zeta) - \nu^{-2}B_\nu(\zeta)C_\nu(\zeta) = 1. \tag{2.23}$$

By substituting $C_\nu(\zeta)$, $D_\nu(\zeta)$ of (2.9) into (2.23) we again obtain (2.21).

The system in (2.19) is equivalent to a (4 by 4)-system of first order equations, admitting four independent solutions. The solution $\{A, A', B, B'\}$ that we need satisfies initial conditions at, say, $\zeta = 0$. Exact initial values of A, A', B, B' at $\zeta = 0$ can be obtained from (2.1). They involve values of the Airy functions (and the derivatives thereof) at the origin, and $J_\nu(\nu), J_\nu'(\nu), Y_\nu(\nu), Y_\nu'(\nu)$. In a numerical scheme for solving the system (2.19) these initial values are needed, up to a certain accuracy.

2.3. *Values of the coefficients at the turning point*

It is convenient to collect some information from the literature on the initial values at the turning point $\zeta = 0, z = 1$ of system (2.19), because these values give insight in recursion relations discussed later. From [1, p. 368] we obtain

$$\begin{aligned}
J_\nu(\nu) &= \nu^{-1/3}2^{1/3}\mathrm{Ai}(0)S(\nu) + \nu^{-5/3}2^{2/3}\mathrm{Ai}'(0)T(\nu),\\
Y_\nu(\nu) &= -\nu^{-1/3}2^{1/3}\mathrm{Bi}(0)S(\nu) - \nu^{-5/3}2^{2/3}\mathrm{Bi}'(0)T(\nu),\\
J_\nu'(\nu) &= -\nu^{-2/3}2^{2/3}\mathrm{Ai}'(0)U(\nu) - \nu^{-4/3}2^{1/3}\mathrm{Ai}(0)V(\nu),\\
Y_\nu'(\nu) &= \nu^{-2/3}2^{2/3}\mathrm{Bi}'(0)U(\nu) + \nu^{-4/3}2^{1/3}\mathrm{Bi}(0)V(\nu),
\end{aligned} \tag{2.24}$$

in which S, T, U, V denote functions having the following asymptotic expansions:

$$S(\nu) \sim \sum_{n=0}^{\infty} \frac{\alpha_n}{\nu^{2n}}, \quad \alpha_0 = 1,\ \alpha_1 = -\tfrac{1}{225},$$

$$T(\nu) \sim \sum_{n=0}^{\infty} \frac{\beta_n}{\nu^{2n}}, \quad \beta_0 = \tfrac{1}{70},\ \beta_1 = -\tfrac{1213}{1023750},$$

$$
\begin{aligned}
U(\nu) &\sim \sum_{n=0}^{\infty} \frac{\gamma_n}{\nu^{2n}}, \quad \gamma_0 = 1, \ \gamma_1 = \tfrac{23}{3150}, \\
V(\nu) &\sim \sum_{n=0}^{\infty} \frac{\delta_n}{\nu^{2n}}, \quad \delta_0 = \tfrac{1}{5}, \ \delta_1 = -\tfrac{947}{346500}.
\end{aligned} \tag{2.25}
$$

From the Wronskian in (2.22) it follows that

$$
T(\nu)V(\nu) = \nu^2\big[S(\nu)U(\nu) - 1\big]. \tag{2.26}
$$

The function $\phi(\zeta)$ defined in (2.4) has expansion $\phi(\zeta) = 2^{1/3} + \frac{1}{5}\zeta + \mathrm{O}(\zeta^2)$. It follows from (2.1), (2.8) and (2.22) that

$$
\begin{aligned}
A_\nu(0) &= S(\nu), & A'_\nu(0) &= 2^{-1/3}\big[V(\nu) - \tfrac{1}{5}S(\nu)\big], \\
B_\nu(0) &= 2^{1/3}T(\nu), & B'_\nu(0) &= -\tfrac{1}{5}T(\nu) + \nu^2\big[U(\nu) - S(\nu)\big].
\end{aligned} \tag{2.27}
$$

It is easily verified that $A'_\nu(0) = \mathrm{O}(\nu^{-2})$, $B'_\nu(0) = \frac{2}{225} + \mathrm{O}(\nu^{-2})$, as $\nu \to \infty$. Observe that leading terms in $V(\nu) - \frac{1}{5}S(\nu)$ and $U(\nu) - S(\nu)$ cancel each other.

3. Expansions of the coefficients

Our purpose is to describe an algorithm for computing the Bessel functions $J_\nu(\nu z), Y_\nu(\nu z)$ and their derivatives in the neighborhood of the turning point $z = 1$ for large values of the parameter ν. The powerful Airy-type expansions can be used for this purpose. We do not consider the evaluation of the Airy functions here, because several algorithms are available for these functions; see the overview in [4].

We concentrate on the evaluation of the functions $A_\nu, B_\nu, C_\nu, D_\nu$ introduced in section 2 for ζ near the origin, say for $|\zeta| \leqslant 1$. For real values of ζ this gives an interval in the z-domain around $z = 1$, that is, $[0.39, 1.98]$. A straightforward method is based on using Maclaurin series expansions of the quantities involved in powers of ζ.

The singular points of the functions $z(\zeta), \psi(\zeta), \phi(\zeta), \widehat{\phi}(\zeta), \chi(\zeta)$ and those of the coefficients of the asymptotic expansions occur at

$$
\zeta^{\pm} = \left(\tfrac{3}{2}\pi\right)^{2/3} \mathrm{e}^{\pm i\pi/3} \tag{3.1}
$$

(see [7, p. 421]). These points correspond with the $z = \mathrm{e}^{\mp i\pi}$. It follows that the radius of convergence of the Maclaurin series of these quantities equals $2.81\ldots$. In this section we give the expansions and mention the values of the early coefficients.

It is convenient to start with an expansion of z in powers of ζ. We obtain from (2.3)

$$
\zeta z^2 = \left(1 - z^2\right)\left(\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right)^2
$$

Table 1
First terms of the Maclaurin expansions of the functions $z(\zeta), \psi(\zeta), \phi(\zeta), \widehat{\phi}(\zeta), \chi(\zeta)$; the parameter η is given by $\eta = 2^{-1/3}\zeta$.

$z(\zeta) =$	$\sum_{n=0}^{\infty} z_n \eta^n$	$= \left[1 - \eta + \frac{3}{10}\eta^2 + \frac{1}{350}\eta^3 - \frac{479}{63000}\eta^4 + \cdots\right],$
$\psi(\zeta) =$	$2^{1/3}\sum_{n=0}^{\infty} \psi_n \eta^n$	$= 2^{1/3}\left[\frac{1}{70} + \frac{2}{75}\eta + \frac{138}{13475}\eta^2 - \frac{296}{73125}\eta^3 - \frac{38464}{7074375}\eta^4 + \cdots\right],$
$\phi(\zeta) =$	$2^{1/3}\sum_{n=0}^{\infty} \phi_n \eta^n$	$= 2^{1/3}\left[1 + \frac{1}{5}\eta + \frac{9}{350}\eta^2 - \frac{89}{15750}\eta^3 - \frac{4547}{1155000}\eta^4 + \cdots\right],$
$\widehat{\phi}(\zeta) =$	$2^{2/3}\sum_{n=0}^{\infty} \widehat{\phi}_n \eta^n$	$= 2^{2/3}\left[1 + \frac{4}{5}\eta + \frac{18}{35}\eta^2 + \frac{88}{315}\eta^3 + \frac{79586}{606375}\eta^4 + \cdots\right],$
$\chi(\zeta) =$	$2^{-1/3}\sum_{n=0}^{\infty} \chi_n \eta^n$	$= 2^{-1/3}\left[\frac{1}{5} + \frac{2}{175}\eta - \frac{64}{2625}\eta^2 - \frac{30424}{3031875}\eta^3 + \frac{173648}{197071875}\eta^4 + \cdots\right].$

and substitute $z = 1 + z_1\zeta + \cdots$. This gives $z_1^3 = -1/2$. Using the relations in (2.3) we obtain the correct branch: $z_1 = -2^{-1/3}$. We write

$$\zeta = 2^{1/3}\eta, \tag{3.2}$$

and we obtain in a straightforward way the expansions shown in table 1.

Next we consider the coefficients a_s, b_s that occur in (2.2). We expand

$$a_s(\zeta) = \sum_{t=0}^{\infty} a_s^t \eta^t, \qquad b_s(\zeta) = 2^{1/3}\sum_{t=0}^{\infty} b_s^t \eta^t, \tag{3.3}$$

where η is given in (3.2). The coefficients a_s^t, b_s^t are rational numbers. We know that $a_0(\zeta) = 1$. Substituting the expansions in (2.14) we can obtain recursion relations for the coefficients a_s^t, b_s^t. It follows that

$$\begin{aligned} 2(2t+1)b_s^t &= 2\sum_{r=0}^{t} \psi_r a_s^{t-r} - (t+1)(t+2)a_s^{t+2}, \\ 2(t+1)a_{s+1}^{t+1} &= 2\sum_{r=0}^{t} \psi_r b_s^{t-r} - (t+1)(t+2)b_s^{t+2}. \end{aligned} \tag{3.4}$$

The relations are used for fixed $s \geqslant 0$, while $t = 0, 1, 2, \ldots$. When $s = 0$ the first relation gives $b_0^t = \psi_t/(2t+1)$, $t = 0, 1, 2, \ldots$. We observe that the second relation does not give a value for a_1^0. The same problem occurs for all values of s.

To find a_s^0 we can use (2.21). By substituting the expansions of (2.2), it follows that for $s = 0, 1, 2, \ldots$

$$\sum_{r=0}^{s+1} a_r(\zeta)a_{s+1-r}(\zeta) + \sum_{r=0}^{s}\left[a_r(\zeta)b_{s-r}'(\zeta) - a_r'(\zeta)b_{s-r}(\zeta) - \zeta b_r(\zeta)b_{s-r}(\zeta)\right] = 0. \tag{3.5}$$

Putting $\zeta = 0$ yields

$$2a_{s+1}^0 = -\sum_{r=1}^{s} a_r^0 a_{s+1-r}^0 - \sum_{r=0}^{s}\left[a_r^0 b_{s-r}^1 - a_r^1 b_{s-r}^0\right], \quad s = 0, 1, 2, \ldots. \tag{3.6}$$

Table 2
First terms of the Maclaurin expansions of the coefficients $a_s(\zeta), b_s(\zeta)$; cf. (3.3); the parameter η is given by $\eta = 2^{-1/3}\zeta$.

$$a_0(\zeta) = 1,$$
$$a_1(\zeta) = -\frac{1}{225} - \frac{71}{38500}\eta + \frac{82}{73125}\eta^2 + \frac{5246}{3898125}\eta^3 + \frac{185728}{478603125}\eta^4 + \cdots,$$
$$a_2(\zeta) = \frac{151439}{218295000} + \frac{68401}{147262500}\eta - \frac{1796498167}{4193689500000}\eta^2 - \frac{583721053}{830718281250}\eta^3 + \cdots,$$
$$a_3(\zeta) = -\frac{887278009}{2504935125000} - \frac{3032321618951}{9708942993750000}\eta + \cdots,$$
$$b_0(\zeta) = 2^{1/3}\left[\frac{1}{70} + \frac{2}{225}\eta + \frac{138}{67375}\eta^2 - \frac{296}{511875}\eta^3 - \frac{38464}{63669375}\eta^4 + \cdots\right],$$
$$b_1(\zeta) = 2^{1/3}\left[-\frac{1213}{1023750} - \frac{3757}{2695000}\eta - \frac{3225661}{6700443750}\eta^2 + \frac{90454643}{336992906250}\eta^3 + \cdots\right],$$
$$b_2(\zeta) = 2^{1/3}\left[\frac{16542537833}{37743205500000} + \frac{115773498223}{162820783125000}\eta + \frac{548511920915149}{1721719224225000000}\eta^2 + \cdots\right],$$
$$b_3(\zeta) = 2^{1/3}\left[-\frac{430990563936859253}{568167343994250000000} - \frac{3191320338955050557}{7777535495585625000000}\eta + \cdots\right].$$

In this way we obtain the expansions shown in table 2. Expansions for the coefficients c_s, d_s are not really needed, because these quantities follow from the relations in (2.11), if expansions for the functions in the right-hand sides of (2.11) are available.

3.1. Numerical experiments

We have used the expansions in (3.3) for $|\zeta| \leqslant 1$ for obtaining values of $a_s(\zeta)/\nu^{2s}$, $b_s(\zeta)/\nu^{2s+4/3}$ for $s = 0, 1, 2, \ldots, 5$ with absolute accuracy of 10^{-20} if $\nu \geqslant 100$. We have used the series in (3.3) with terms up to $t = 45 - 6s$. The evaluated series in (3.3) and those of the derivatives have been used to check the Wronskian relation in (2.21) for a set of values of ζ on the unit circle. The results are shown in the third column of table 3. The same errors have been obtained by calculating the quantities in (2.21) by using the explicit representations in (2.5), and those of the derivatives by using (2.12) and (2.13).

In the fourth column the relative errors in $b_0(\zeta)$ are shown, where we compared values of $b_0(\zeta)$ obtained by using explicit representations and by using the Maclaurin expansions. In the final column we give the results for $a_1(\zeta)$. The results for $b_0(\zeta)$ and $a_1(\zeta)$ are accurate enough for use in (2.1) and (2.2) in order to obtain about 20 decimal digits accuracy on the unit circle in the ζ-plane for J_ν, Y_ν. Also in the higher order coefficients $b_s(\zeta)$ and $a_{s+1}(\zeta)$, $s \geqslant 1$, less accuracy is needed because the negative powers of ν in the series in (2.2).

We have computed the explicit representations of $a_s(\zeta), b_s(\zeta)$ by using the computer algebra facilities of Maple. To obtain numerical values we have computed the quantities with the Maple parameter Digits set equal to 20. In this way we expected to have a fair comparison with the Maclaurin expansions, although it is quite easy to obtain higher accuracy in Maple by setting Digits equal to larger values. Also, the coefficients used in the Maclaurin expansions (3.3) are converted to 20 decimal digits in the computations.

Table 3
Relative errors of the relation (2.21) for values of ζ on the upper part of the unit circle, $\zeta = e^{n\pi i/16}$, by comparing the results obtained by using the explicit representations in (2.5) and the expansions (3.3). In the fourth column the relative errors of $b_0(\zeta)$ obtained by both methods are shown; the same for $a_1(\zeta)$ in the final column.

n	$\theta = n\pi/16$	error in (2.21)	error in $b_0(\zeta)$	error in $a_1(\zeta)$
0	$0\pi/16$	0.14×10^{-20}	0.54×10^{-18}	0.45×10^{-19}
1	$1\pi/16$	0.71×10^{-20}	0.84×10^{-18}	0.10×10^{-16}
2	$2\pi/16$	0.64×10^{-20}	0.24×10^{-18}	0.41×10^{-17}
3	$3\pi/16$	0.23×10^{-20}	0.90×10^{-18}	0.56×10^{-17}
4	$4\pi/16$	0.20×10^{-19}	0.22×10^{-18}	0.12×10^{-16}
5	$5\pi/16$	0.51×10^{-20}	0.85×10^{-18}	0.18×10^{-16}
6	$6\pi/16$	0.26×10^{-20}	0.39×10^{-18}	0.17×10^{-16}
7	$7\pi/16$	0.13×10^{-19}	0.54×10^{-18}	0.18×10^{-16}
8	$8\pi/16$	0.28×10^{-20}	0.59×10^{-19}	0.19×10^{-16}
9	$9\pi/16$	0.26×10^{-20}	0.55×10^{-18}	0.23×10^{-16}
10	$10\pi/16$	0.14×10^{-19}	0.58×10^{-18}	0.22×10^{-16}
11	$11\pi/16$	0.47×10^{-20}	0.89×10^{-18}	0.32×10^{-16}
12	$12\pi/16$	0.33×10^{-20}	0.11×10^{-17}	0.23×10^{-16}
13	$13\pi/16$	0.20×10^{-19}	0.47×10^{-18}	0.30×10^{-16}
14	$14\pi/16$	0.12×10^{-21}	0.29×10^{-18}	0.33×10^{-16}
15	$15\pi/16$	0.20×10^{-19}	0.98×10^{-19}	0.32×10^{-16}
16	$16\pi/16$	0.90×10^{-20}	0.92×10^{-19}	0.35×10^{-16}

We conclude from these experiments that, for checking the Wronskian relation in (2.21) with the required precision of 20 decimal digits, and for real values of ν larger than 100, we can use the boundary of the unit disk in the ζ-plane to decide about using Maclaurin expansions of the coefficients a_s, b_s or their explicit representations.

Exact values of the coefficients needed in this algorithm (the first few values are shown in tables 1 and 2) are available from the author upon request.

4. Evaluation of the functions $A_\nu(\zeta)$, $B_\nu(\zeta)$ by iteration

We now concentrate on solving the system of differential equations in (2.19) by using analytical techniques. Instead of expanding the coefficients a_s, b_s of the asymptotic series we expand the functions $A_\nu(\zeta), B_\nu(\zeta)$ in Maclaurin series. As remarked earlier, the singular points of these functions occur at $\zeta^\pm = \left(\frac{3}{2}\pi\right)^{2/3} e^{\pm i\pi/3}$, and the radius of convergence of the series of $A_\nu(\zeta)$ and $B_\nu(\zeta)$ in powers of ζ equals $2.81\ldots$.

We expand

$$A_\nu(\zeta) = \sum_{n=0}^{\infty} f_n(\nu)\zeta^n, \qquad B_\nu(\zeta) = \sum_{n=0}^{\infty} g_n(\nu)\zeta^n, \qquad \psi(\zeta) = \sum_{n=0}^{\infty} h_n\zeta^n. \tag{4.1}$$

The coefficients $f_0, f_1, \ldots, g_0, g_1, \ldots$ are to be determined, with the first elements given in (2.27), while the coefficients h_n are known. The first few h_n follow from (3.2) and table 1:

$$h_0 = \tfrac{1}{70}2^{1/3}, \qquad h_1 = \tfrac{2}{75}, \qquad h_2 = \tfrac{69}{13475}2^{2/3}, \qquad h_3 = \tfrac{148}{73125}2^{1/3}.$$

Upon substituting the expansions into (2.19), we obtain for $n = 0, 1, 2, \ldots$ the recursion relations

$$\begin{aligned} (n+2)(n+1)f_{n+2} + (2n+1)g_n &= \rho_n, \qquad \rho_n = \sum_{k=0}^{n} h_k f_{n-k}, \\ (n+2)(n+1)g_{n+2} + 2\nu^2(n+1)f_{n+1} &= \sigma_n, \qquad \sigma_n = \sum_{k=0}^{n} h_k g_{n-k}. \end{aligned} \tag{4.2}$$

We have already observed that cancellation occurs in the representations in (2.27). All evaluations based on the above recursions for computing higher coefficients f_n, g_n from lower coefficients suffer from cancellations. That is, the recursion relations cannot be used in the *forward direction*. In particular when ν is large the recursions in (4.2) are not stable in the forward direction.

To show what happens, we give a few details on the first recursion. Take $n = 0$, then we obtain, using $f_0 = A_\nu(0), g_0 = B_\nu(0)$ and (2.27),

$$\begin{aligned} 2f_2 &= f_0 h_0 - g_0 = 2^{1/3}\left[\tfrac{1}{70}S(\nu) - T(\nu)\right], \\ 2g_2 &= g_0 h_0 - 2\nu^2 f_1 = 2^{2/3}\left[\tfrac{1}{70}T(\nu) - \nu^2\left\{V(\nu) - \tfrac{1}{5}S(\nu)\right\}\right]. \end{aligned}$$

We see from (2.25) that $f_2 = \mathrm{O}(\nu^{-2})$, $g_2 = \mathrm{O}(1)$, as $\nu \to \infty$, whereas the quantities used to compute f_2 are of order O(1). Also, the term with ν^2 in g_2 is of lower order in the final result. Further use of the recursion makes things worse. In fact, in further steps more and more early terms in the asymptotic expansions of combinations of $S(\nu)$, $T(\nu)$, $U(\nu)$ and $V(\nu)$ are subtracted.

This unstable pulling down of asymptotic series suggests to use the recursion in (4.2) in the backward direction. When we try to use (4.2) in the backward direction, for instance with false starting values f_N, g_N for some large integer N, a complication arises because of the terms ρ_n, σ_n on the right-hand sides of (4.2). All terms ρ_n, σ_n contain f_k, g_k for $k = 0, 1, 2, \ldots, n$. Hence, recursion in the backward direction is not possible at all. A way out is to consider ρ_n, σ_n as known quantities, and to treat (4.2) as inhomogeneous difference equations.

4.1. Solving (2.19) by iteration

A first step in this approach will be to solve the system (2.19) by iteration. That is, we choose an appropriate pair of functions F_0, G_0, and define two sequences of functions $\{F_m\}$, $\{G_m\}$ by writing for $m = 1, 2, 3, \ldots$:

$$F_m'' + 2\zeta G_m' + G_m = \psi(\zeta)F_{m-1}, \qquad G_m'' + 2\nu^2 F_m' = \psi(\zeta)G_{m-1}. \tag{4.3}$$

To study this iterative process we need to know the solutions of the homogeneous equations, that is, of the system

$$\begin{aligned} F_m'' + 2\zeta G_m' + G_m &= 0, \\ G_m'' + 2\nu^2 F_m' &= 0. \end{aligned} \tag{4.4}$$

One solution is $F = 1$, $G = 0$. Other solutions of (4.4) follow by eliminating F'' in the first equation by differentiating the second one. The result is

$$G''' - 4\nu^2\zeta G' - 2\nu^2 G = 0, \tag{4.5}$$

with solutions products of Airy functions (see [1, p. 448]):

$$\mathrm{Ai}^2(t), \qquad \mathrm{Ai}(t)\mathrm{Bi}(t), \qquad \mathrm{Bi}^2(t), \qquad t = \nu^{2/3}\zeta. \tag{4.6}$$

The F-solutions of the homogeneous equations (4.4) follow from integrating the second line in (4.4). Knowing these four linearly independent solutions we can construct solutions F, G of the inhomogeneous equations corresponding to (4.4), that is, the system (2.19), by using the variation of constants formula, and eventually by constructing Volterra integral equations defining the solutions A, B of (2.19). For details we refer to section 4.3 below.

4.2. *Solving (4.3) by backward recursion*

We rewrite (4.2) in backward form:

$$\begin{aligned} f_n &= \frac{1}{2\nu^2}\left[\tfrac{1}{n}\sigma_{n-1} - (n+1)g_{n+1}\right], \\ g_{n-1} &= \frac{1}{2n-1}\left[\rho_{n-1} - n(n+1)f_{n+1}\right], \end{aligned} \tag{4.7}$$

where $n \geqslant 1$. The coefficients are assumed to belong to the functions $F_m(\zeta), G_m(\zeta)$ of the iteration process described by (4.3), while the coefficients ρ_{n-1}, σ_{n-1} are assumed to be known, and contain Maclaurin coefficients of $F_{m-1}(\zeta)$, $G_{m-1}(\zeta)$ and $\psi(\zeta)$. Observe that (4.7) does not define f_0. After having computed $f_1, f_2, \ldots, g_0, g_1, g_2, \ldots$ by the backward recursion process, we compute f_0 from the Wronskian (2.21):

$$f_0 = \frac{-g_1 + \sqrt{g_1^2 + 4\nu^2(\nu^2 + f_1 g_0)}}{2\nu^2}, \tag{4.8}$$

where the +sign of the square root is taken because of the known behaviour of $F_\nu(0)$ when ν is large; see (2.27).

We give a few steps in the iteration and backward recursion process. Let us start the iterations (4.3) with constant (F_0, G_0) (constant with respect to ζ and ν). The obvious constant choice of (F_0, G_0) is $(1, h_0)$; see (2.27). We use the four coefficients of $\psi(\zeta)$ shown after (4.1) for constructing the ρ and σ coefficients in the right-hand sides of (4.7). We have

$$\rho_n = h_n, \qquad \sigma_n = h_0 h_n, \qquad n = 0,1,2,3, \qquad \rho_n = \sigma_n = 0, \qquad n \geqslant 4.$$

Then the first iteration gives

$$\begin{aligned} f_4 &= \tfrac{1}{8} h_0 h_3 \nu^{-2}, & g_3 &= \tfrac{1}{7} h_3, \\ f_3 &= \tfrac{1}{6} h_0 h_2 \nu^{-2}, & g_2 &= \tfrac{1}{5} h_2 - \tfrac{3}{10} h_0 h_3 \nu^{-2}, \\ f_2 &= \tfrac{1}{2}\left(\tfrac{1}{2} h_0 h_1 - \tfrac{3}{7} h_3\right)\nu^{-2}, & g_1 &= \tfrac{1}{3} h_1 - \tfrac{1}{3} h_0 h_2 \nu^{-2}, \\ f_1 &= \tfrac{1}{2}\left(h_0^2 - \tfrac{2}{5} h_2 + \tfrac{3}{5} h_0 h_3 \nu^{-2}\right)\nu^{-2}, & g_0 &= h_0 - \left(\tfrac{1}{2} h_0 h_1 - \tfrac{3}{7} h_3\right)\nu^{-2}, \end{aligned} \tag{4.9}$$

while f_0 is computed by using (4.8). Expanding the result for f_0 we find

$$f_0 = 1 - \frac{1}{225}\nu^{-2} + \mathrm{O}\left(\nu^{-4}\right),$$

which agrees with the first two terms of the asymptotic expansion of $T(\nu)$ given in (2.25). Also, the first terms of the asymptotic expansions of g_0, f_1, g_1 agree with the first terms of the expansions following from (2.25) and (2.27). When more coefficients h_k and more iterations are used, the further iterates F_m, G_m have Maclaurin coefficients f_n, g_n of which the asymptotic expansions with respect to ν are converging to the actual asymptotic expansions of f_n, g_n. In particular, the asymptotic expansions of f_0, g_0 coincide more and more with those following from (2.27). Of course, it is not our goal to obtain the asymptotic expansions of the coefficients f_n, g_n, but this illustrates the analytical nature of the algorithm.

The numerical problem in using the recursions in (4.2) in the forward direction is the influence of dominant solutions of the homogeneous equations of (4.2) (that is, the equations obtained by taking $\rho_n = \sigma_n = 0$). The dominant solutions are the Maclaurin coefficients of the functions given in (4.6), as functions of ζ. The coefficients grow as ν becomes large. The minimal solution is given by $f_0 = 1$ and $f_{n+1} = g_n = 0$, $n \geqslant 0$. From the above observations we infer that the solutions of the inhomogeneous equations (4.2) cannot contain dominant solutions of the homogeneous equations. This explains the unstable character of the forward recursions based on (4.2) and the stable character of the recursion based on the backward form in (4.7). More details on these phenomena can be found in [11].

4.3. *On the convergence of the iterations in (4.3)*

To obtain a solution of the system (2.19), which we write in the form

$$\begin{aligned} F'' + 2\zeta G' + G - \psi(\zeta) F &= 0, \\ G'' + 2\nu^2 F' - \psi(\zeta) G &= 0, \end{aligned}$$

we introduced the iterations in (4.3). We can write this in matrix form

$$\begin{aligned} y'(\zeta) &= A(\zeta)y(\zeta) \;+ B(\zeta)y(\zeta),\\ y'_m(\zeta) &= A(\zeta)y_m(\zeta) + B(\zeta)y_{m-1}(\zeta), \end{aligned}$$

where $m = 1, 2, 3, \ldots$ and

$$y(\zeta) = \begin{pmatrix} F \\ G \\ U \\ V \end{pmatrix}, \qquad A(\zeta) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2\zeta \\ 0 & 0 & -2\nu^2 & 0 \end{pmatrix},$$

$$B(\zeta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \psi(\zeta) & 0 & 0 & 0 \\ 0 & \psi(\zeta) & 0 & 0 \end{pmatrix},$$

and $F' = U$, $G' = V$.

We take $\zeta_0 \in \mathbb{R}$ and $\Phi(\zeta)$ as a fundamental matrix of the system $y'(\zeta) = A(\zeta)y(\zeta)$, i.e., the columns of the matrix are composed of linearly independent solutions of the homogeneous equation $y'(\zeta) = A(\zeta)y(\zeta)$, with $\Phi(\zeta_0) = I_4$. As we remarked earlier (cf. (4.5), (4.6)), we can indeed find these solutions, and because the solutions are linearly independent and analytic, it follows that $\Phi(\zeta)$ is invertible in $\mathbb{C}$ and that $\Phi^{-1}(\zeta)$ has the same regularity as $\Phi(\zeta)$. Applying the variation of constants formula, we find

$$\begin{aligned} y(\zeta) &= \Phi(\zeta)y(\zeta_0) \;+ \Phi(\zeta)\int_{\zeta_0}^{\zeta} \Phi^{-1}(t)B(t)y(t)\,\mathrm{d}t,\\ y_m(\zeta) &= \Phi(\zeta)y_m(\zeta_0) + \Phi(\zeta)\int_{\zeta_0}^{\zeta} \Phi^{-1}(t)B(t)y_{m-1}(t)\,\mathrm{d}t. \end{aligned}$$

So, if we take $y_m(\zeta_0) = y(\zeta_0)$,

$$y_m(\zeta) - y(\zeta) = \Phi(\zeta)y_m(\zeta_0) + \Phi(\zeta)\int_{\zeta_0}^{\zeta} \Phi^{-1}(t)B(t)\big[y_{m-1}(t) - y(t)\big]\,\mathrm{d}t.$$

We consider a matrix norm $|\cdot|$ in $\mathbb{C}_4$; let $\|\cdot\|$ be its subordinated matrix norm, and let $\alpha > \zeta_0$. We are going to prove that $y_m \to y$ for $\zeta \in [\zeta_0, \alpha]$. Take, for $\zeta \in [\zeta_0, \alpha]$,

$$\begin{aligned} M &= \sup\big(\|\Phi(\zeta)\|, \|\Phi^{-1}(\zeta)\|\big),\\ P &= \sup\|B\|,\\ \rho &= \sup\big\{\big|y_1(\zeta) - y(\zeta)\big|\big\}. \end{aligned}$$

Then

$$\big|y_2(\zeta) - y(\zeta)\big| \leqslant M^2P\int_{\zeta_0}^{\zeta} \big|y_1(s) - y(s)\big|\,\mathrm{d}s \leqslant \rho M^2P(\zeta - \zeta_0),$$

$$\left|y_3(\zeta) - y(\zeta)\right| \leqslant M^2 P \int_{\zeta_0}^{\zeta} \left|y_2(s) - y(s)\right| \mathrm{d}s \leqslant \rho M^2 P M^2 P \int_{\zeta_0}^{\zeta} (s - \zeta_0)\,\mathrm{d}s$$
$$= \rho\left(M^2 P\right)^2 \frac{(\zeta - \zeta_0)^2}{2}.$$

Continuing this procedure, we finally obtain

$$\left|y_m(\zeta) - y(\zeta)\right| \leqslant \rho \frac{\left(M^2 P\right)^{m-1}}{(m-1)!} (\zeta - \zeta_0)^{m-1},$$

which tends to zero as $m \to \infty$. The proof for $\alpha < \zeta_0$, $\zeta \in [\alpha, \zeta_0]$ and for complex values of ζ, ζ_0 is similar.

4.4. *Numerical experiments*

For numerical applications information is needed about the growth of the coefficients f_n, g_n. Since the Maclaurin series in (4.1) have a radius of convergence equal to $2.81\ldots$, for all values of ν, the size of the coefficients f_n, g_n is comparable with that of h_n. It depends also on the size of $|\zeta|$ how many coefficients f_n, g_n are needed in (4.1). When $|\zeta| = 1$ we need about 45 terms in the Maclaurin series in (4.1) in order to obtain an accuracy of about 20 decimal digits. The ζ-interval $[-1, 1]$ corresponds to the z-interval $[0.39, 1.98]$. When z is outside this interval many other efficient algorithms are available for the computation of $J_\nu(\nu z), Y_\nu(\nu z)$.

We have computed successive iterates of Maclaurin coefficients f_n, g_n defined in (4.1) for different values of ν. To give the algorithm some relevant starting values we have used approximations for f_0, g_0 based on (2.27), with a few terms of $S(\nu), T(\nu)$ of (2.25). Furthermore we have taken $g_n = h_n/(2n+1)$, $n \geqslant 1$, which choice is based on taking $A = 1$ in the first line of (2.19), and integrating the resulting relation $(\sqrt{\zeta}\,B)' = \psi/(2\sqrt{\zeta})$.

During each iteration we start the backward recursions with $f_n = g_{n-1} = 0$, $n \geqslant 46$, and we compute $f_{45}, g_{44}, f_{44}, g_{43}, \ldots$ by using (4.4). We use h_k, $k = 0, 1, \ldots, 45$ and we recompute the coefficients ρ_k, σ_k, $k = 0, 1, 2, \ldots, 45$, using (4.2) with values f_k, g_k obtained in the previous iteration. In table 4 we show the relative errors in the values $f_0, g_0, f_5, g_5, f_{10}, g_{10}$, when compared with more accurate values f_0^a, etc. Computations are done with extended precision (machine accuracy about 10^{-19}). The accurate values are obtained by applying the backward recursion by using 10 iterations. We also give the relative error in the Wronskian relation (2.21) at $\zeta = 1$ during each iteration.

From table 4 we conclude that for $\nu = 5$ we can already obtain an accuracy of 10^{-10} in the Wronskian after two iterations; further iterations improve the results. For larger values of ν the algorithm is very efficient.

Small values of ν do not cause problems in the numerical algorithms published in the literature. When using the above algorithm for computing the Bessel functions, also the Airy functions and the functions $\phi(\zeta)$ and $\zeta(z)$, all occurring in (2.1), are

Table 4
Relative errors during five iterations (i) of $f_0, g_0, f_5, g_5, f_{10}, g_{10}$ compared with more accurate values f_0^a, etc. The final column shows the relative error in the Wronskian (2.21) at $\zeta = 1$.

i	$\lvert f_0 - f_0^a \rvert$	$\lvert g_0 - g_0^a \rvert$	$\lvert f_5 - f_5^a \rvert$	$\lvert g_5 - g_5^a \rvert$	$\lvert f_{10} - f_{10}^a \rvert$	$\lvert g_{10} - g_{10}^a \rvert$	Wronskian
				$\nu = 5$			
1	6.11e–09	1.76e–06	1.12e–03	6.14e–04	3.38e–03	1.55e–03	4.36e–08
2	4.54e–12	1.03e–08	6.14e–06	8.33e–07	2.22e–05	4.42e–06	2.05e–10
3	2.56e–15	1.60e–11	1.52e–08	8.48e–10	5.47e–08	8.24e–09	3.29e–13
4	1.21e–17	1.83e–14	6.40e–12	5.25e–13	1.86e–11	2.95e–11	6.72e–16
5	0.00e–00	4.19e–17	4.64e–14	4.04e–16	2.58e–14	1.26e–14	2.23e–18
				$\nu = 10$			
1	4.24e–10	1.14e–07	2.90e–04	1.63e–04	8.92e–04	4.30e–04	2.76e–09
2	8.50e–14	8.17e–10	1.84e–06	5.76e–08	6.91e–06	3.16e–07	1.64e–11
3	1.45e–17	3.20e–13	1.10e–09	9.06e–11	4.27e–09	9.28e–10	6.64e–15
4	1.08e–19	9.89e–17	1.74e–12	1.25e–15	4.79e–12	6.87e–13	8.07e–18
5	0.00e–00	1.88e–19	1.35e–15	6.74e–17	1.58e–15	4.32e–16	3.44e–19
				$\nu = 25$			
1	1.12e–11	2.94e–09	4.70e–05	2.66e–05	1.45e–04	7.09e–05	7.10e–11
2	3.66e–16	2.22e–11	3.09e–07	1.52e–09	1.18e–06	8.44e–09	4.47e–13
3	0.00e–00	1.40e–15	2.95e–11	2.66e–12	1.17e–10	2.76e–11	2.91e–17
4	0.00e–00	0.00e–00	5.63e–14	7.25e–18	1.61e–13	3.17e–15	1.75e–19
5	0.00e–00	0.00e–00	6.45e–18	3.40e–19	8.67e–18	3.14e–18	1.76e–19
				$\nu = 50$			
1	7.02e–13	1.84e–10	1.18e–05	6.66e–06	3.64e–05	1.78e–05	4.44e–12
2	5.75e–18	1.40e–12	7.79e–08	9.53e–11	2.97e–07	5.31e–10	2.82e–14
3	0.00e–00	2.21e–17	1.85e–12	1.69e–13	7.39e–12	1.76e–12	5.57e–19
4	0.00e–00	0.00e–00	3.63e–15	1.70e–19	1.05e–14	5.05e–17	1.26e–19
5	0.00e–00	0.00e–00	2.80e–19	0.00e–00	2.01e–19	8.94e–20	1.26e–19

needed. So the method based on the evaluation of $A_\nu(\zeta), B_\nu(\zeta)$ may not be faster than existing algorithms when ν is less than 100, say.

5. Discussion and conclusions

We have described two methods for evaluating Airy-type asymptotic expansions for the Bessel functions $J_\nu(\nu z), Y_\nu(\nu z)$ (and for their derivatives) near the turning point $z = 1$. For the Hankel functions the same methods are applicable.

The first method described in section 2 for evaluating the asymptotic series of the Airy-type expansions requires the storage of many pre-computed coefficients. When these are available evaluation of the asymptotic series near the turning point $z = 1$, $\zeta = 0$ is rather straightforward and efficient. One has to be sure whether for a given value of ν and the required precision enough terms are available in the asymptotic series. The accuracy in the evaluation of $A_\nu, B_\nu, C_\nu, D_\nu$ can be checked by using the relation in (2.23).

In the second method of section 3 one needs only the storage of the coefficients h_n of the Maclaurin series for ψ; see (4.1) and (2.15). An algorithm based on this method can reach any desired accuracy (already for moderate values of ν), if enough coefficients h_n are available. The two components in the algorithm:

- the iteration of the pair of functions $\{F_m, G_m\}$ (see (4.4)),
- the backward recursion scheme for the coefficients f_n, g_n (see (4.2)),

are both numerically stable, and become more efficient as ν increases. The computer experiments shown in table 4 indicate that this method is very promising.

The methods of this paper can be used for Airy-type asymptotic expansions for other special functions. We mention as interesting cases parabolic cylinder functions, Coulomb wave functions, and other members of the class of Whittaker functions. To stay in the class of Bessel functions, we mention the modified Bessel function of the third kind $K_{i\nu}(z)$ of imaginary order, which plays an important role in the diffraction theory of pulses and in the study of certain hydrodynamical studies. Moreover, this function is the kernel of the Lebedev transform. The same functions $A_\nu, B_\nu, C_\nu, D_\nu$ can be used for this case; see [3] for many details. It seems that there is no published code for the numerical evaluation of the function $K_{i\nu}(x)$ that covers the case of large parameters.

It is of interest to compare the algorithms of Amos [2] and Matviyenko [5] with our algorithms, but we expect to return to this in future publications, when we also want to consider the modified Bessel functions with purely imaginary order (cf. also [9], where contours of steepest descents are given for $K_{i\nu}(x)$).

Acknowledgment

The author wishes to thank Dr. Juan Campos Rodriguez of the University of Granada (Spain) for his help in the proof given in section 4.3, and the referee for helpful comments on the first version of the paper.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Nat. Bur. Standards Appl. Series 55 (U.S. Government Printing Office, Washington, 1964).

[2] D. Amos, Algorithm 644, A portable package for Bessel functions of a complex argument and nonnegative order, ACM Trans. Math. Software 12 (1986) 265–273.

[3] T.M. Dunster, Bessel functions of purely imaginary order, with an application to second-order linear differential equations having a large parameter, SIAM J. Math. Anal. 21 (1990) 995–1018.

[4] D.W. Lozier and F.W.J. Olver, Numerical evaluation of special functions, in: *Mathematics of Computation, 1943–1993: A Half-Century of Computational Mathematics*, ed. W. Gautschi, AMS, Proc. Symp. Appl. Math. 48 (1994) pp. 79–125.

[5] G. Matviyenko, On the evaluation of Bessel functions, Appl. Comp. Harm. Anal. 1 (1993) 116–135.

[6] A.B. Olde Daalhuis and N.M. Temme, Uniform Airy type expansions of integrals, SIAM J. Math. Anal. 25 (1994) 304–321.

[7] F.W.J. Olver, *Asymptotics and Special Functions* (Academic Press, New York, 1974, reprinted A.K. Peters, 1997).

[8] N.M. Temme, On the computation of the incomplete gamma functions for large values of the parameters, in: *Algorithms for Approximation, Proc. of the IMA Conference on Algorithms for the Approximation of Functions and Data*, eds. J.C. Mason and M.G. Cox (Clarendon, Oxford, 1987) pp. 479–489.

[9] N.M. Temme, Steepest descent paths for integrals defining the modified Bessel functions of imaginary order, Methods Appl. Anal. 1 (1994) 14–24.

[10] N.M. Temme, *Special Functions: An Introduction to the Classical Functions of Mathematical Physics* (Wiley, New York, 1996).

[11] J. Wimp, *Computation with Recurrence Relations* (Pitman, Boston, 1984).